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at these bases, indeed the angle *KLK* to the angle *KMK*, and the angle *LKK* to the angle *MKK*. Therefore equal also are the remainders *NKM* and *HKL*. Wherefore, since the sides *NK*, *KM* of the triangle *NKM* are equal in the same way to the sides *HK*, *KL* of the triangle *HKL*, equal also will be (from the same Eu. I. 4) the bases *NM*, *HL*, the angles *KNM*, *KHL*, and finally the angles *KMN*, *KLH*. But in the preceding triangles are already proved equal the angles *KLK*, *KMK*. Therefore the whole angle *NMK* is equal to the whole angle *HLK*.

Wherefore, since all angles at the points *K* are right, it follows manifestly all four angles together of the quadrilateral *KNMK* are equal to all four angles together of the quadrilateral *KHLK*.

But since the two angles together at the points *N* and *M* in the quadrilateral *KNMK* are greater, in hypothesis of acute angle, than the two angles together (from Cor. after P. XVI) at the points *D* and *H* in the quadrilateral *NDHM*, or the quadrilateral *KDHK*, the consequence thence is, that (the common right angles at the points *K* being added) the four angles together of the quadrilateral *KNMK*, or the quadrilateral *KHLK* are greater (in hypothesis of acute angle) than the four angles together of the quadrilateral *KDHK*.

Quod erat demonstrandum.

COROLLARY.

But it ought here opportunely to be observed, nothing will fail in the argument made, although the angle at the point *L* is assumed right, together with hypothesis of acute angle. For still that common perpendicular *LK* would be less (from Cor. I. after II of this) than the other perpendicular *HK*, from which therefore still a portion *MK* could be assumed equal to the aforesaid *LK*.

Which standing, it follows that no hindrance can intervene.

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

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[Continued from April Number.]

PRIMITIVE GROUPS OF TWO, THREE, AND FOUR, LETTERS.

Since all of these must contain substitutions of the form a_1a_2 or of the form $a_1a_2a_3$ they must all contain the symmetric group. The following is therefore a complete list:

Degree.	Order.	Group.
2	2	(*ab*)
3	3	(*abc*)
	6	(*abc*)all
4	12	(*abcd*)pos
	24	(*abcd*)all.

PRIMITIVE GROUPS OF FIVE LETTERS.

All the transitive groups of this degree must be primitive and there must be one regular group, viz:

(1) (*abcde*).

The lowest order of any other possible primitive group is 10. Such a primitive group would contain five subgroups of the form (*ab.cd*) and therefore four substitutions of degree 5. All the substitutions of degree 5 whose powers do not contain a substitution of the type a_1a_2 are of the type $a_1a_2a_3a_4a_5$. Hence all the substitutions of degree 5 of a primitive group which is not the symmetric group must be of the given type.

If a primitive group of order 10 exists we may therefore assume that it contains

(*abcde*)

and some substitutions of the form *ab.cd*. These substitutions are all equal to

(*abcde*)*S*

where *S* is any one among them. They therefore transform the substitutions of (*abcde*) into the same power. This cannot be the first power for a substitution consisting of a single cycle can be transformed into the first power only by its own powers. If we represent this power by α and observe that the product of two of these supstitutions is equal to some substitution in (*abcde*) we have*

$$\alpha^2 \equiv 1 \pmod 5), \; (1 < \alpha < 5).$$

Since this has only one solution it follows that there is only one group of order 10. We may find the substitutions by writing the fourth power of *abcde* under *abcde*, thus,

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*If s_1 and s_2 transform s into s^α we have

$$s_1^{-1}ss_1 = s^\alpha$$

$$s_2^{-1}s_1^{-1}ss_1s_2 = s_2^{-1}s^\alpha s_2 = s_2^{-1}ss_2 . s_2^{-1}ss_2 \ldots\ldots\ldots\ldots \alpha \text{ times}$$

$$= s^\alpha . s^\alpha . s^\alpha \ldots\ldots\ldots\ldots\ldots \alpha \text{ times}$$

$$= s^{\alpha^2}$$

$$\begin{array}{ccccc} abcde & abcde & abcde & abcde & abcde \\ aedcb & baedc & cbaed & dcbae & edcba. \end{array}$$

The required substitutions are

$$be.cd \qquad ab.ce \qquad ac.de \qquad ad.bc \qquad ae.bd$$

[We might evidently have obtained all of these by multiplying one into $(abcde)$]. Hence the group of order 10 is

$$(abcde)(ab.ce)=(abcde)_{10}. \tag{2}$$

All the substitutions that transform $(abcde)$ into itself form a group. There are five substitutions that transform the substitutions of $(abcde)$ into their first power, therefore there must be five that transform them into each of their other powers. We thus obtain a group of order 20 which is generated by $abcde$ and some substitution $bced$ which transforms this into its second power. We have therefore

$$(abcde)(bced)=(abcde)_{20}. \tag{3}$$

There cannot be more than one of this order because each would have to contain five conjugate subgroups of one of the two types

$$(abcd)_4, \ (abcd)$$

and therefore only one subgroup (necessarily self conjugate) of the type

$$(abcde).$$

This may be supposed to be the same in all of the groups; but there is only one set of twenty substitutions that transform this into itself. The groups are therefore identical.

For all the other possible orders the subgroups of degree 4 would contain either a substitution of the type ab or one of the type abc. Hence all the other primitive groups are the alternating and the symmetric group. The following is a complete list of the primitive groups of degree five.

Order.	Group.
5	$(abcde)$
10	$(abcde)_{10}$
20	$(abcde)_{20}$
60	$(abcde)$pos
120	$(abcde)$all

These groups could also have been found in the following manner, without employing the groups of a lower degree. We know that there is one group of each of the three classes—regular, alternating and symmetric. We know also that the order of each of the other primitive groups exceeds five and that they do not contain any substitutions of either of the two types

$$ab \qquad abc$$

Hence they can contain only substitutions of the fourth and fifth degrees together with unity.

Since the average number of letters in all the substitutions of these groups must be four each group can contain only four substitutions of the fifth degree. The only type of substitutions of the fifth degree which can be used is

$$abcde.$$

All these primitive groups may therefore be supposed to contain

$$(abcde)$$

as a self-conjugate subgroup and to be subgroups of the group of order 20 which contains all the substitutions that transform $(abcde)$ into itself.

Any negative substitution of this group together with $(abcde)$ generates the entire group, the only subgroup besides the group itself and $(abcde)$ must therefore consist of the positive substitutions of the group. Hence there are only two primitive groups of degree five in addition to the regular, alternating, and symmetric groups. The generating substitutions of these groups are evident.

PRIMITIVE GROUPS OF SIX LETTERS.

There is no regular group. If there were a group of order 30 it would contain 24 substitutions of the type *abcde* and five substitutions of degree six. These five substitutions would generate a regular group; for only one of them could replace a given letter by any required letter since there are four of the form *abcde* which perform this operation, and therefore the product of any two must be of degree six or it must be unity.

[To be Continued.]